$$0 < \mu < 0{,}010913\ldots;\ 0{,}016376\ldots < \mu < \mu_1 = 0{,}024293\ldots \quad (8.5)$$
$$\mu_1 < \mu < 0{,}038520\ldots$$

system (5.3) has no other solution than the trivial. Hence we have the following theorem.

Theorem 5. The triangular libration points of the three-dimensional restricted three-body problem is Lyapunov stable for all values of the parameter μ from the interval (8.3).

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ESTIMATE OF THE STABILITY OF A DYNAMIC SYSTEM ON THE BASIS OF THE QUASISTATIONARITY PRINCIPLE *

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The following problem is formulated and solved: in what cases, and on what basis for examining the stability of the stationary solution of a "quasistationary" system can we judge the stability of the stationary solution of the initial system? The theorems which formulate the necessary and sufficient conditions of the stability are proved. It is shown how the results obtained can be used to examine the thermal stability of a chemical reactor.

1. Suppose it is required to examine the stability of the stationary state of a dynamic system. When using Lyapunov's first method this problem reduces (if we do not consider special cases) to the problem of verifying the stability of the zeroth solution of the linearized system. We will assume that the latter can be represented in the form

$$\frac{dy}{dt} = Ay + Bz, \quad \frac{dz}{dt} = Cy - Dz; \quad y \in R^m, \quad z \in R^l \quad (1.1)$$

We will also introduce the notation $x = (y_1, \ldots, y_m, z_1, \ldots, z_l)^T$, $m + l = n$, where the index T denotes transposition.

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We will call the specified matrix stable (strongly stable), if all $\operatorname{Re}\lambda_i \leqslant 0$ ($\operatorname{Re}\lambda_i < 0$), and unstable, if i exists, for which $\operatorname{Re}\lambda_i > 0$, where λ_i are the eigenvalues of the matrix.

Suppose

$$F = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$$

The problem consists of obtaining the conditions of the stability of the matrix F. Henceforth we will assume everywhere that D and F are non-degenerate.

Expressing z from the equation $Cy + Dz = 0$ and substituting it into the first Eq.(1.1), we will obtain a linearized "quasistationary" system

$$dy/dt = (A - BD^{-1}C)\, y = (A + BK)\, y = A^*y \tag{1.2}$$

We will call the matrix A^* quasistationary.

We will distinguish the case $m = 1$, which in practice is of independent importance.

We will formulate the following problem: under which conditions can we draw a conclusion from the stability or instability of the stationary state of the quasistationary system about the stability or instability of the stationary state of the initial system?

By virtue of the results connected with A.N. Tikhonov's theorem /1/, it is natural to expect that if the stationary state of the z-system

$$dz/dt = Dz \tag{1.3}$$

is asymptotically stable and the system fairly rapidly relaxes to it, then the stability or instability of the stationary state of the quasistationary system determines the stability or instability of the stationary state of the initial system. It is interesting, however, to obtain working estimates enabling us to draw a conclusion about the stability or instability of the matrix F from the stability or instability of the matrix A^*. This paper is aimed at obtaining those estimates. Note that, as will follow from the results obtained below, an estimate of the rate of relaxation of the z-system is not needed in a large class of cases to obtain the necessary conditions of stability.

The formulation of part of the results of this paper is given in /2/.

2. We will first obtain the necessary conditions of stability.

Lemma 1. The parity of the number of real positive eigenvalues (bearing in mind their multiplicity) of the matrix F is the product of the corresponding parities for matrices A^* and D, determined using the rules of Boolean algebra: P X P= O X O = P, P X O = O X P = O (P denotes parity, O denotes odd parity).

Proof. We will use the following representation (see problem 2.4, ch.1 in /3/):

$$\det F = \prod_{i=1}^{n} \lambda_i = \det A^* \cdot \det D \tag{2.1}$$

where λ_i are the eigenvalues of the matrix F. From (2.1) it follows that: $\lambda_1 \ldots \lambda_n = (\lambda_1^* \ldots \lambda_m^*) \times (\mu_1 \ldots \mu_l)$, where λ_j^*, μ_k are the eigenvalues of the matrices A^* and D. All $\lambda_i \neq 0$ and, consequently, all $\lambda_j^* \neq 0$. Suppose the numbers of positive real eigenvalues of the matrices F, A^* and D equal k_1, k_2 and k_3 respectively. We have (since the number of complex eigenvalues of the real matrix is even)

$$\operatorname{sgn}(\lambda_1 \ldots \lambda_n) = (-1)^{n-k_1}, \quad \operatorname{sgn}(\lambda_1^* \ldots \lambda_m^*) = (-1)^{m-k_2}$$
$$\operatorname{sgn}(\mu_1 \ldots \mu_l) = (-1)^{l-k_3}$$

Consequently, $(-1)^{k_1} = (-1)^{k_2+k_3}$ and the lemma is proved.

The following theorem is a simple corollary of Lemma 1:

Theorem 1. Suppose the matrix D is stable. Then the instability of matrix F follows from the instability of the quasistationary matrix A^* when there is an odd number of positive real eigenvalues. When $m = 1$ the instability of F follows from the instability of A^*.

Theorem 1 formulates the necessary conditions of stability. It is essential that there is an additional requirement about the odd number of positive real eigenavlues of the matrix A^* in the formulation of the theorem: in general the instability of the matrix F does not follow from the instability of the matrix A^*. A corresponding counter-example can be constructed for the case $n = 3$, $m = 2$.

3. Let us proceed to the sufficient conditions of stability. When deriving these conditions we will confine ourselves to the case $m = 1$. For this case B is a row-vector, K is a column-vector and A and A^* are numbers.

Suppose further that $z^* = Ky = -D^{-1}Cy$, $\Delta z = z - z^*$, $\delta z = \Delta z / |z^*|$ when $z^* \neq 0$, where $|z|$ is the Euclidean norm z. We will assume that $C \neq 0$ (therefore, $K \neq 0$). After transformations we obtain

$$dy/dt = (A^* + \operatorname{sgn}(y) \mid K \mid B\delta z)\, y = \bar{A} y \tag{3.1}$$
$$d\Delta z/dt = D\Delta z - \bar{A} z^* \tag{3.2}$$
$$d\delta z/dt = D\delta z - \bar{A}\,(\delta z + L)\ (L = \operatorname{sgn}(y)\, K/ \mid K \mid) \tag{3.3}$$

Lemma 2. Suppose when $0 \leqslant t \leqslant t_1$

$$x(t) = R(t) + \int_0^t S(t, \tau)\, \Phi(x(\tau))\, d\tau; \quad x, R, \Phi \in R^\beta \tag{3.4}$$

where S is a $\beta \times \beta$-matrix, and the functions $R(\cdot)$, $S(\cdot, \cdot)$ and $\Phi(\cdot)$ are continuously differentiable. Suppose also

$$r(t) = |R(t)|, \quad s = \max_{0 \leqslant \tau_1, \tau_2 \leqslant t_1} |S(\tau_1, \tau_2)|$$
$$|S| = \max_{|x|=1} |Sx|, \quad |\Phi(x)| \leqslant \varphi(|x|)$$

and $\varphi(\cdot)$ is a non-decreasing function. Then, if the equation

$$\alpha(t) = r(t) + \int_0^t s\varphi(\alpha(\tau))\, d\tau \tag{3.5}$$

has a solution in $[0, t_1]$, the following estimate holds:

$$|x(t)| \leqslant \alpha(t)$$

Proof. Suppose $\alpha_\varepsilon(t)$ satisfies Eq.(3.5) with $r(t)$ replaced by $r_\varepsilon(t) = r(t) + \varepsilon$, where ε is a small positive number. Suppose t' is the minimum t for which $\alpha_\varepsilon(t) = |x(t)|$. Then $t' > 0$ and $|x(t)| < \alpha_\varepsilon(t)$ when $0 \leqslant t < t'$. But by virtue of the latter, and bearing in mind (3.4), we have

$$|x(t')| < r_\varepsilon(t') + \int_0^{t'} s\varphi(\alpha_\varepsilon(\theta))\, d\theta = \alpha_\varepsilon(t')$$

which leads to a contradition. Therefore $|x(t)| \leqslant \alpha_\varepsilon(t)$ for all $t \in [0, t_1]$. Letting $\varepsilon \to 0$, we will obtain the statement of the lemma.

Lemma 3. Suppose $\alpha_2(t)$ is a scalar function which satisfies the equation

$$\alpha_2(t) = r(t) + \int_0^t f_2(\alpha_2(\tau))\, d\tau \tag{3.6}$$

in $[0, t_1]$ and suppose $f_2(\alpha)$ is a non-decreasing function α, whilst $0 \leqslant f_1(\alpha) \leqslant f_2(\alpha)$ and the functions $r(t)$, $f_1(\alpha)$, $f_2(\alpha)$ are continuously differentiable. Then the solution of the equation

$$\alpha_1(t) = r(t) + \int_0^t f_1(\alpha_1(\tau))\, d\tau \tag{3.7}$$

exists in $[0, t_1]$ and for all $t \in [0, t_1]$ the following inequality holds:

$$\alpha_1(t) \leqslant \alpha_2(t)$$

Proof. Suppose first that $f_1(\alpha) < f_2(\alpha)$ for all α. In some neighbourhood $[0, t')$ $\alpha_1(t)$ exists and $\alpha_1(t) < \alpha_2(t)$ when $t > 0$. It is obvious that $\alpha_1(t)$ can continue as long as the inequality $\alpha_1(t) < \alpha_2(t)$ holds. Suppose t'' is the minimum $t > t'$ for which $\alpha_1(t) = \alpha_2(t)$. But $\alpha_1(t'') < \alpha_2(t'')$ follows from the conditions of the lemma, which implies a contradiction.

To prove the lemma in the general case we will introduce $f_{2,\varepsilon}(\alpha) = f_2(\alpha) + \varepsilon$ and then proceed to the limit as $\varepsilon \to 0$.

Lemma 4. Suppose when $t \in [0, t_1]$

$$\alpha(t) = r(t) + \int_0^t s\alpha(\tau)\, d\tau \tag{3.8}$$

and $s \geqslant 0$, $r(t) \geqslant 0$ and the function $r(\cdot)$ is continuously differentiable. Then

$$\alpha(t) \leqslant r(t) + r_{\max}(e^{st} - 1); \quad r_{\max} = \max_{0 \leqslant \tau \leqslant t_1} r(\tau) \tag{3.9}$$

The statement of Lemma 4 follows from the known representation of the solution of Eq. (3.8)

$$\alpha(t) = e^{st}\alpha_0 + \int_0^t e^{s(t-\tau)} r^{\cdot}(\tau)\, d\tau \tag{3.10}$$

Suppose further that $A^* < 0$. Consider the solution of Eq.(3.3) in the interval $[0, t_1]$, in which $|y(t)| > 0$ (in this interval $\delta z(t)$ remains finite). By virtue of well-known results of the theory of ordinary differential equations /4/ we have

$$\delta z(t) = F_D(t)\,\delta z_0 - \int_0^t F_D(t-\theta)\,\bar{A}(\delta z(\theta))(\delta z(\theta) + L)\,d\theta; \qquad F_D(t) = z(t)\,z(0)^{-1} \tag{3.11}$$

where $z(t)$ is a fundamental matrix of the solutions of the uniform system (1.3), $\delta z_0 = \delta z(0)$.

Suppose matrix D is strongly stable. We will introduce the following parameters which characterize the relaxation processes of a uniform system to a stationary state:

$$t_D = \max_{z_0} \min_{t>0} t: \; |F_D(t) z_0| = {}^1/_2 |z_0| \tag{3.12}$$

$$q = \max_{0 \leqslant t \leqslant t_D} |F_D(t) z_0|, \qquad |z_0| = |z(0)| = 1 \tag{3.13}$$

The parameter t_D is the minimum time of the guaranteed half decrease of the norm of the initial phase vector z_0 (here considered as the relaxation time of the z-system), and q is the maximum "buildup" of the phase vector when relaxing in accordance with Eq.(1.3) to the state $z = 0$ during the time t_D. By virtue of the strong stability of matrix D the quantities t_D and q are finite, and $q \geqslant 1$.

To facilitate further understanding we will first outline the following considerations and logical transitions.

Consider the motion for which $y > 0$. At some instant $t_1 \leqslant t_D$ the equation $|F_D(t_1)\,\delta z_0| = {}^1/_2 |\delta z_0|$ occurs. We will find the conditions at $c_0 > 0$. for which any $\delta z(t)$ with $|\delta z_0| \leqslant c_0$ is limited in $[0, t_1]$ and

$$\left|\int_0^{t_1} F(t-\theta)\,\bar{A}(\delta z(\theta))(\delta z(\theta) - L)\,d\theta\right| \leqslant {}^1/_2 c_0$$

When these conditions hold the quantity $\delta z(t)$ with $\delta z_0 \leqslant c_0$ is bounded over the whole interval $[0, +\infty)$ and, at least, periodically falls within the sphere V_{c_0} of radius c_0 with its centre at the origin of coordinates. At the same time it appears that for all $\delta z_0 \in V_{c_0}$, with the exception of δz_0 from some subspace of the space E^l, $\delta z(t) \to \delta z_*$ as $t \to +\infty$ and $\delta z_* \in V_{c_0}$ is the stationary point of system (3.3). Hence it follows that $\bar{A}(\delta z_*)$ is an eigenvalue of the matrix F and $\bar{A}(\delta z_*) = \max_i \operatorname{Re} \lambda_i$. We will supplement the conditions obtained with the condition which guarantees that the inequalities $\bar{A}(\delta z) < 0$ hold when $\delta z \in V_{c_0}$. Then the asymptotic stability of the zeroth solution of system (1.1) follows from the validity of the conditions introduced, and estimates for the eigenvalue with a maximum real part are obtained.

We shall present some calculations. Applying Lemma 2 to Eq.(3.11), we obtain: $|\delta z(t)| \leqslant \alpha_1(t)$ for $t \leqslant t_1 = t(\delta z_0) \leqslant t_D$, where

$$\alpha_1(t) = d(t) + q|A^*|t + q\int_0^t (|K||B|\alpha_1^2(\theta) + |K||B|\alpha_1(\theta) + |A^*|\alpha_1(\theta))\,d\theta; \quad d(t) = |F_D(t)\,\delta z_0| \tag{3.14}$$

(assuming that Eq.(3.14) has a solution). By virtue of Lemma 3, if $\alpha(t)$ satisfies equation

$$\alpha(t) = d(t) + q|A^*|t + \int_0^t p\alpha(\theta)\,d\theta \tag{3.15}$$
$$p = q(2|B||K| + |A^*|)$$

in $[0, t_1)$ and the inequality

$$\alpha(t) \leqslant 1, \quad t \in [0, t_1] \tag{3.16}$$

holds, then Eq.(3.14) has a solution and $|\delta z(t)| \leqslant \alpha_1(t) \leqslant \alpha(t)$.

Suppose further that $|\delta z_0| \leqslant c_0$, $c_0 > 0$. Using Lemma 4 as applied to Eq.(3.15), we obtain the estimate

$$\alpha(t) \leqslant d(t) + c_0 q(e^{pt} - 1) + qt|A^*|e^{pt} \tag{3.17}$$

It follows from (3.17) that inequality (3.16) will hold if

$$c_0 \leqslant q^{-1}e^{-pt_1} - |A^*|t_1 \tag{3.18}$$

Suppose at the instant t_1

$$|\delta z(t_1)| \leqslant c_0 \tag{3.19}$$

Since by the definition of t_1 the condition $d(t_1) = {}^1/_2|\delta z_0| \leqslant {}^1/_2 c_0$ holds, then inequality (3.19) holds if

$$c_0 q\,(e^{pt_1} - 1) + qt_1\,|A^*|\,e^{pt_1} \leqslant {}^1/_2 c_0 \tag{3.20}$$

From (3.20) and the condition $c_0 > 0$ it follows that

$$c_0 \geqslant \frac{qt_1\,|A^*|\,e^{pt_1}}{{}^1/_2 - q\,(e^{pt_1} - 1)}, \quad q\,(e^{pt_1} - 1) \leqslant \frac{1}{2} \tag{3.21}$$

Suppose the following inequalities hold

$$0 < c_1 = \frac{qt_D\,|A^*|\,e^{pt_D}}{{}^1/_2 - q\,(e^{pt_D} - 1)} < \frac{1}{q}\,e^{-pt_D} - |A^*|\,t_D = c_2 \tag{3.22}$$

Then when $c_1 \leqslant c_0 \leqslant c_2$ inequalities (3.18) and (3.21) hold.

By considering the calculations in reverse order, we can verify that when conditions (3.22) hold, if $c_1 \leqslant c_0 \leqslant c_2$ and $|\delta z_0| \leqslant c_0$, then $\delta z\,(t)$ over the whole interval $[0, +\infty)$ remains all the time in a sphere of the unit radius and, at least periodically, falls within the sphere V_{c_0}.

We shall determine

$$R_0 = \frac{|A^*|}{|K|\,|B|}\,(K = -D^{-1}C,\ A^* = A + BK) \tag{3.23}$$

Since (when $y > 0$) $\bar{A}\,(\delta z) = A^* + |K|\,B\delta z$, the following condition holds:

$$\begin{aligned} &\bar{A}\,(\delta z) < 0 \ \text{ when } \ |\delta z| < R_0 \\ &\bar{A}\,(\delta z) = 0 \ \text{ when } \ \delta z = R_0 B^T / |B| \end{aligned} \tag{3.24}$$

We will require the following inequality to hold:

$$c_1 < R_0 \tag{3.25}$$

Then for $\delta z\,(t)$ the condition $\bar{A}\,(\delta z) < 0$ periodically holds.

Theorem 2. Suppose the matrix D is strongly stable, $C \neq 0$, $A^* < 0$ and the following inequality holds:

$$t_D\,|A^*| < \chi\,(R_0, q) \tag{3.26}$$

where t_D, q and R_0 are determined using conditions (3.12), (3.13) and (3.23), and $\chi\,(R_0, q)$ is the only root (for the variable χ) of the equation

$$\begin{aligned} P\,(\chi, R_0, q) = {} & \min\left(R_0, \frac{1}{q}\exp\left(-\chi q\left(\frac{2}{R_0} + 1\right)\right) - \chi\right) - \\ & q\chi\exp\left(\chi q\left(\frac{2}{R_0} + 1\right)\right)\left[\frac{1}{2} - q\left(\exp\left(\chi q\left(\frac{2}{R_0} + 1\right)\right) - 1\right)\right]^{-1} = 0 \end{aligned} \tag{3.27}$$

in the interval

$$0 < \chi < \frac{\ln\,(1/(2q) + 1)}{2/R_0 + 1} = \chi_1 \tag{3.28}$$

Then the matrix F is strongly stable, whilst the quantity λ_{i*} ($\mathrm{Re}\,\lambda_{i*} = \max_i \mathrm{Re}\,\lambda_i$) is real and satisfies the estimates

$$A^* - |K|\,|B|\,c_1 \leqslant \lambda_{i*} \leqslant A^* + |K|\,|B|\,c_1 < 0 \tag{3.29}$$

Proof. The existence and uniqueness of the root of Eq.(3.27) follows from the continuity and monotony of $P\,(\chi, R_0, q)$, if we bear in mind that $P\,(0, R_0, q) = \min\,(R_0, 1)$, $P\,(\chi_1, R_0, q) = -\infty$. Further, as we can verify

$$\begin{aligned} c_1 &= \frac{qt_D\,|A^*|\exp\,(t_D\,|A^*|\,(2/R_0 + 1))}{{}^1/_2 - q\,(\exp\,(t_D\,|A^*|\,(2/R_0 + 1)) - 1)} \\ c_2 &= \frac{1}{q}\exp\,(-t_D\,|A^*|\,(2/R_0 + 1)) - t_D\,|A^*| \end{aligned} \tag{3.30}$$

Therefore $P\,(t_D\,|A^*|, R_0, q) = \min\,(R_0, c_2) - c_1$ and condition (3.26) is the same as the condition

$$0 < c_1 < \min\,(R_0, c_2) \tag{3.31}$$

Suppose the eigenvalues of the matrix F are different. Consider the motion of the point x of system (1.1) with initial conditions which satisfy the relations

$$y\,(0) > 0,\ \ \delta z_0 = (z\,(0) - Ky\,(0))\,(\,|K|\,y\,(0)) \leqslant c_0 = c_1$$

and the motion $\delta z(t)$ corresponding to it. Using what has earlier been proved, the quantity $\delta z(t)$ remains bounded when $0 \leqslant t < +\infty$ and, therefore, $y(t) > 0$. (If for some t we have $y(t) = 0$, then for this t we will have $|\delta z(t)| = +\infty$.) We will call these motions separate. Suppose ξ is a unit eigenvector, satisfying λ_{i*} ($\mathrm{Re}\,\lambda_{i*} = \max_i \mathrm{Re}\,\lambda_i$). The quantity $\xi_1 \neq 0$, since when $\xi_1 = 0$ separate motions exist, for which $\delta z(t)$ is not a bounded function. Therefore, λ_{i*} and ξ are real (otherwise separate motions, for which $y(t)$ for fairly large t would complete the oscillations around zero, would exist).

We can assume that $\xi_1 > 0$. For all the separate motions, with the exception of those which are completed at some hyperplane determined by eigenvectors which differ from ξ, $x(t)$ $|x(t)| \to \xi$ as $t \to +\infty$. For those motions

$$\delta z(t) \to \delta z_* = \frac{\eta - \xi_1 K}{\xi_1 |K|}$$

and this means that in the sphere V_{c_0} system (3.3) has a stationary point. But $\bar{A}(\delta z_*) = \lambda_{i*}$ (see (3.1)). Therefore, bearing in mind (3.25), we conclude that the estimates (3.29) hold and the matrix F is strongly stable.

To prove the theorem in the general case we will intorudce a family of matrices $F(\sigma)$: $F(\sigma)$ depends in a continuous way on σ, $F(0) = F$, $F(\sigma)$ has different eigenvalues when $\sigma \neq 0$. As can be shown, q, t_D, A^*, K and B are continuous functions of the parameter σ (while $F(\sigma)$ remains strongly stable). Therefore for fairly small σ the inequalities (3.31) hold and, therefore, estimates (3.29) hold. Proceeding to the limit as $\sigma \to 0$, we will obtain the statement of the theorem for $F(0) = F$. The theorem is proved.

The quantity $1/|A^*|$ characterizes the inertia of the quasistationary system, and t_D characterizes the inertia of the z-system. Therefore inequality (3.26) shows that from the inequality $A^* < 0$ we can draw a conclusion about the stability of matrix F in those cases when the inertia of the z-system is fairly small compared with that of the quasistationary system.

Note that the function $\chi(R_0, q)$ decreases as q increases and R_0 decreases.

Note 1. In the formulation of Theorem 2 it is assumed that $C \neq 0$ and, therefore, $K \neq 0$. If $C = 0$ then, as we can show, the eigenvalues of the matrix F include $A = A^*$ and the eigenvalues of the matrix D. Therefore for this case the strong stability of F under the condition of strong stability of D and $A^* < 0$ is trivial.

Note 2. The statement of Theorem 2 remains valid of the parameters t_D and q are replaced by their upper estimates, and the parameter R_0 is replaced by its lower estimate. The validity of this statement follows from the fact that (see (3.30)) c_1 is an increasing function and c_2 is a decreasing function of the parameters t_D and q.

To use Theorem 2 we need to know t_D and q or the upper estimates of these parameters. The Lyapunov-function method is an effective way of obtaining these estimates.

Theorem 3. Suppose $V(z)$ is a positive definite uniform Lyapunov function of the z-system (1.3) (thereby for $V(z)$ the condition $V(az) = aV(z)$ holds when $a > 0$), which satisfies the conditions

$$\frac{\max |z| \text{ when } V(z) = 1}{\min |z| \text{ when } V(z) = 1} = q' \tag{3.32}$$

$$dV/dt \leqslant eV \quad (e < 0) \tag{3.33}$$

Then q' and $t_D' = \ln(2q')/|e|$ are upper estimates for q and t_D.

The statement of the theorem for q' is obvious. The statement for t_D' follows from Lemma 3.

Note 3. Note that for the asymptotic stability of the zeroth solution of (1.3) a uniform Lyapunov function always exists, satisfying the inequality (3.33) with $e = (\mathrm{Re}\,\mu)^{\bullet}_{\max}$, where $(\mathrm{Re}\,\mu)_{\max} = \max_i \mathrm{Re}\,\mu_i$ (μ_i are eigenvalues of the matrix D). In fact, confining ourselves to the non-singular case, we will assume that all μ_i are different. We will reduce D to a diagonal form using a non-degenerate linear transformation of S. We can show, by direct verification, that $V = (Sz, \overline{Sz})^{1/2}$ is a uniform Lyapunov function for which inequality (3.33) holds with $e = (\mathrm{Re}\,\mu)_{\max}$.

It follows from the above that condition (3.26) in Theorem 2 can also be written in the following form:

$$\frac{|A^*| \ln(2q^*)}{|(\mathrm{Re}\,\mu)_{\max}|} < \chi(R_0, q^*) \tag{3.34}$$

where q^* is the parameter which satisfies condition (3.32) for Lyapunov's function of the z-system, for which inequality (3.33) holds with $\epsilon = (\mathrm{Re}\mu)_{max}$.

When there are no direct analytical results in a number of cases the upper estimates for t_D and q can be found from the experimental data or on the basis of a combined approach which combines the analytical and experimental results.

4. In the theory of a thermal explosion the stability criterion, defined by the sign of the derivative dQ/dT [5, 6], where $Q = Q_1 - Q_2$, Q_1 is the amount of heat dissipated in the reactor and Q_2 is the amount withdrawn, is well known. The stationary states for which $dQ/dT < 0$, are stable, and the states with $dQ/dT > 0$ are unstable.

This result assumes, strictly speaking, that the dynamics of the process are determined by one differential equation. Tests of the analytical basis of this criterion as applied to a wider class of cases were only made for second-order systems (for an ideal-mixing reactor, whose dynamic process is determined by the temperature concentration). It was shown that for these systems the criterion of the sign of dQ/dT generally gives the necessary - but not the sufficient - conditions of stability /7/. This criterion gives the sufficient conditons of stability when the relaxation time of the concentration model is substantially less than that of the thermal model /7, 8/. On the other hand, numerous calculational experiments show that the criterion of the sign of dQ/dT obviously holds in an extremely wide class of cases.

We will use the result of paras. 2 and 3 to investigate the thermal stability of the stationary state of an ideal-mixing exothermic reactor. We will consider the following dynamic reactor model:

$$H_T \frac{dT}{dt} = Q(T, C) = Q_1(T, C) - Q_2(T, C) = \sum_i (-\Delta H_i) r_i(T, C) - \frac{C_p(T - T_0)}{\tau} - K_T S(T - T_c) \tag{4.1}$$

$$H_c \frac{dC_j}{dt} = w_j(T, C) - \frac{C_j - C_{j0}}{\tau}, \quad j = 1, 2, \ldots, l \tag{4.2}$$

$$w_j = \sum_i \gamma_{ij} r_i \tag{4.3}$$

Here C and T are the vector of concentrations and temperature in the reactor, C_{j0} is the concentration of the j-th material at the reactor input, T_0 is the temperature at the reactor input, t is the time, H_T, H_c are the heat and material holding capacities, ΔH_i is the thermal effect of the i-th reaction, C_p is the heat capacity, K_T is the heat transfer coefficient, S is the specific surface of the heat transfer, T_c is the coolant temperature, τ is the conditional contact time, r_i is the rate of the i-th reaction, w_j is the rate of formation of the j-th matter, and (γ_{ij}) is the stoichiometric matrix.

We will call system (4.2) a concentration model with a fixed temperature T.

Suppose (T_s, C_s) is the stationary solution of system (4.1), (4.2). The following theorem directly follows from Theorem 1:

Theorem 4. Suppose the stationary solution C_s of the concentration model (when $T = T_s$) is asymptotically stable. Then the condition

$$dQ/dT\ (T_s, C\ (T_s)) \leqslant 0 \tag{4.4}$$

is the necessary condition of the stability of the solution (T_s, C_s).

In Eq.(4.4) $C(T)$ is the quasistationary concentration, determined by the equations

$$w_j\ (T, C) - (C_j - C_{j0})\ \tau = 0, \quad j = 1, 2, \ldots, l \tag{4.5}$$

Therefore, the criterion of the sign of dQ/dT generally gives the necessary condition of stability, if the stationary solution of the concentration model is asymptotically stable. We stress that this result holds without any demands on the relaxation time of the concentration model.

The asymptotic stability of the concentration model, which is required for the application of Theorem 4, holds in a wide class of cases (this also determines, first of all, the justification of separating the problem of thermal stability from the general problem of reactor stability). At the same time the presence of concentration stability can often be proved globally for a whole class of kinetic relations /9, 10/.

To use Theorem 2 (which formulates the sufficient conditions for stability) it is necessary to have, besides the proof of the asymptotic stability of the stationary solution of the concentration model, upper estimates of the parameters t_D and q also. Lyapunov's-function method provides an effective approach to obtaining these estimates (and to proving the stability of the concentration model). We will write the equations of the concentration model

$$\frac{d\Delta C_j}{dt} = \sum_{k=1}^{l} p_{jk}\Delta C_k - \frac{\Delta C_j}{H_c\tau} \tag{4.6}$$

$$\left(p_{jk} = \frac{1}{H_c}\frac{\partial w_j(T_s, C_s)}{\partial C_k}\right), \quad j = 1, 2, \ldots, l$$

We will term the following system a shortened model:

$$\frac{d\Delta C_j}{dt} = \sum_{k=1}^{l} p_{jk}\Delta C_k, \quad j = 1, 2, \ldots, l \tag{4.7}$$

Theorem 5. Suppose the zero-th solution of system

$$\frac{d\Delta C_j}{dt} = \sum_{k=1}^{l} p_{jk}\Delta C_k - \frac{\Delta C_j}{H_c\tau_1}, \quad \tau_1 > \tau, \ j = 1, 2, \ldots, l \tag{4.8}$$

is asymptotically stable and a uniform positive definite Lyapunov function exists for this system which satisfies estimate (3.32) (with z replaced by ΔC), and inequalities (4.4) hold and

$$\left|\frac{dQ}{dT}(T_s, C(T_s))\right|\frac{\tau H_c \ln(2q')}{1 - \tau/\tau_1} < \chi(R_0, q') \tag{4.9}$$

Then the stationary state of system (4.1)-(4.3) is asymptotically stable.

Proof. It follows from the uniformity of $V(\Delta C)$ that $V(\Delta C) = (\partial V/\partial\Delta C(\Delta C))\Delta C$. We will obtain an estimate for dV/dt by virtue of Eqs.(4.6)

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial\Delta C}P\Delta C - \frac{V}{H_c\tau_1}\right) - \frac{(\tau_1 - \tau)V}{H_c\tau\tau_1} \leqslant -\frac{(\tau_1 - \tau)V}{H_c\tau\tau_1} < 0$$

The statement of the theorem is now a direct corollary of Theorem 3.

Consider the class of kinetic relations which satisfy the relations

$$p_{jj} < 0, \quad p_{jk} \geqslant 0 \quad \text{for} \quad j \neq k, \quad \sum_{j=1}^{l} p_{jk} \leqslant 0, \quad k = 1, 2, \ldots, l \tag{4.10}$$

Note that (4.10) holds if there is no autocatalysis and each rate of reaction in each direction is determined by one "leading" component.

For this class, as is well known and easily verified, $V = \Sigma\,|\Delta C_j|$ is a Lyapunov function of the shortened model (4.7). We can also show that $q' = l^2$. Thus, for this class of kinetic relations condition (4.9) takes the form

$$\left|\frac{dQ}{dT}\right|\tau H_c \ln(2l^2) < \chi(R_0, l^2)$$

Condition (4.9) (assuming that $dQ/dT < 0$) is equivalent to the conditon

$$\frac{T''}{T'} < \chi(R_0, q')\left[\ln(2q')\left(1 - \frac{K_T S\tau}{C_p} - \frac{\tau}{C_p}\frac{dQ_1}{dT}\right)\right]^{-1} \tag{4.11}$$

where $T' = H_T\tau/C_p$, $T'' = H_c\tau$ (these quantities can be considered the time constants of the thermal and concentration models).

For a catalytic reactor often

$$1 + \frac{K_T S\tau}{C_p} - \frac{\tau}{C_p}\frac{dQ_1}{dT} < 2, \quad R_0 \geqslant 1, \quad q' \leqslant 2$$

Then condition (4.11) will hold if $T''/T' < 0.0095$. The latter condition usually holds (in /8/ an example is discussed for which $l = 1$ and $T''/T' = 0.0027$.

In cases when there are no analytical results, experimental data can be included to justify the use of Theorems 1 and 2. Suppose kinetic analyses are carried out in a circulating laboratory reactor which operates in a mode that is close to that of ideal mixing /11/. In such a reactor the heat transfer conditions are usually very good and we can, in conformity with the definition of stability, confine ourselves to the concentration equations which in this case are the same as industrial ones for laboratory apparatus. It therefore follows from the stability of the stationary mode of the laboratory reactor that the concentration model of the industrial process (for equality of τ and C_{j0}) is stable. At the same time the quantities t_D and q can be estimated using dynamic experimental data of the laboratory apparatus.

If the concentration models of the laboratory and industrial reactors are different, the matter is more complex. Here also, however, experiments in the laboratory reactor can in some cases provide useful information for the application of Theorems 1 and 2. For example, for the case of a flowing-circulation laboratory reactor, as we can show, the following formula connecting the characteristic values holds:

$$\frac{1+\tau\gamma H_c p'}{\gamma} - \frac{1-\gamma}{\gamma} g(p') = 1 + \tau H_c p^* \quad (4.12)$$

where p', p^* are characteristic values of p for the transmitting functions of the laboratory and industrial reactors, $g(p)I$ is the transmitting function of the feedback section in the laboratory reactor (I is a unit matrix), and γ is the ratio of the flow at the input to the laboratory reactor to the flow which passes through the reaction volume. It is assumed that the industrial reactor operates for the same C_{j0} as a laboratory reactor, and has $\tau = \tau_0/\gamma$, where τ_0 is the conditional contact time for the reaction volume of the laboratory reactor.

5. The importance of the conditions of stability obtained is determined in the following way.

The conditions of stability obtained on the basis of the quasistationarity principle have a physical meaning in a number of cases and enable us to obtain estimates of a general character. In particular, the criterion of the sign of the derivative dQ/dT in the problem of estimating the thermal stability of the reactor has a physical meaning.

One can also note the calculational simplicity of examining stability on the basis of conditions determined by Theorems land 2, which is particularly useful for a large-size z-system and the necessity for frequent calculations of the stability of the stationary states of the process (with different parameters). The latter occurs, for example, if the process is optimized and the condition of stability figures as one of the optimization limitations /12/.

Finally, the conditions of stability obtained can be used to a certain extent when the dynamic equations of the object are not completely known. To use Theorems 1 and 2 it is sufficient to know the dynamic equations only for y_i. The remaining equations can be stationary if the estimates of the parameters t_D and q are known from the experimental data or by analogy with other processes.

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